

Probabilistic Swarm Guidance using Inhomogeneous Markov Chains

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Abstract—Probabilistic swarm guidance involves designing a Markov chain so that each autonomous agent or robot determines its own trajectory in a statistically independent manner. The swarm converges to the desired formation and the agents repair the formation even if it is externally damaged. In this paper, we present an inhomogeneous Markov chain approach to probabilistic swarm guidance algorithms for minimizing the number of transitions required for achieving the desired formation and then maintaining it. With the help of communication with neighboring agents, each agent estimates the current swarm distribution and computes the tuning parameter which is the Hellinger distance between the current swarm distribution and the desired formation. We design a family of Markov transition matrices for a desired stationary distribution, where the tuning parameter dictates the number of transitions. We discuss methods for handling motion constraints and prove the convergence and the stability guarantees of the proposed algorithms. Finally, we apply these proposed algorithms for guidance and motion planning of swarms of spacecraft in Earth orbit.

I. INTRODUCTION

Small satellites are well-suited for formation flying missions, where multiple satellites operate together in a cluster or predefined geometry to accomplish the task of a single conventionally large satellite. Application of swarms of hundreds to thousands of femtosatellites (100-gram-class satellites) for synthetic aperture applications is discussed in [1]. In this paper, we introduce an inhomogeneous Markov chain approach to develop probabilistic swarm guidance algorithms for constructing and reconfiguring a multi-agent network comprised of a large number of autonomous agents or spacecraft.

Analogous to fluid mechanics, the traditional view of guidance of multi-agent systems is *Lagrangian*, as it deals with an indexed collection of agents [2]–[7]. Note that such deterministic, Lagrangian approaches tend to perform poorly with a large number (100s–1000s) of agents. In this paper, we adopt an *Eulerian* view, as we control the swarm density distribution of a large number of index-free agents over disjoint bins [8], [9]. A centralized approach for controlling the swarm, using optimal control of partial differential equations, is discussed in [10], [11]. Distributed control of the density distribution, using

region-based shape controllers or attraction-repulsion forcing functions, are discussed in [12], [13].

In this paper, *guidance* refers to both motion planning and open-loop control that generate a desired trajectory for each agent [14]. Instead of allocating specific positions to individual agents a priori, a probabilistic guidance algorithm is concerned with achieving the desired swarm distribution across bins [15]–[18]. Each autonomous agent or robot independently determines its transition from one bin to another following a synthesized Markov chain so that the overall swarm converges to the desired formation. Because this Markovian approach automatically fills deficient bins, the resulting algorithm is robust to external damages to the formation. The main limitation of probabilistic guidance algorithm using homogeneous Markov chains, where the Markov matrix M is fixed over time, is that the agents are not allowed to settle down even after the swarm reaches the desired steady-state distribution resulting in significant wastage of control effort (e.g., fuel).

The main contribution of this paper is to develop a probabilistic swarm guidance algorithm using inhomogeneous Markov chains (PSG-IMC) to address this limitation and minimize the number of transitions for achieving and maintaining the formation. Our key concept, which was first presented in [19], is to develop time-varying Markov matrices M_k^j with $\lim_{k \rightarrow \infty} M_k^j = \mathbf{I}$ to ensure that the agents settle down after the desired formation is achieved, thereby minimizing the number of unnecessary transitions across the bins. The inhomogeneous Markov transition matrix for an agent at each time instant depends on the current swarm distribution, the agent's current bin location, the time-varying motion constraints, and the agent's choice of distance-based parameters. We derive proofs of convergence based on analysis of inhomogeneous Markov chains (e.g., [20]–[24]).

It is necessary that each agent communicates with its neighboring agents to estimate the current swarm distribution. Consensus algorithms are studied for formation control, sensor networks, and formation flying applications [2], [25]–[30]. In this paper, by using a decentralized consensus algorithm on probability distributions [31], the agents reach an agreement on the current estimate of the swarm distribution.

Inter-agent collisions are not considered in this paper for concise presentation since a collision-avoidance algorithm within each bin can be combined with the proposed PSG-IMC algorithm (see [32], [33] for details). As an illustrative example, the guidance of swarms of spacecraft orbiting Earth is presented in this paper.

Notation: The *time index* is denoted by a right subscript and

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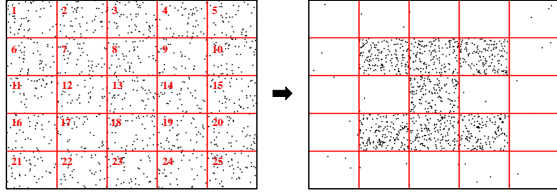


Fig. 1. The PSG-IMC independently determines each agent's trajectory so that the overall swarm converges to the desired formation (here letter "I"), starting from any initial distribution. Here, the state space is partitioned into 25 bins and the desired formation π is given by $\frac{1}{7}[0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0]$.

the *agent index* is denoted by a lower-case right superscript. The symbol $\mathbb{P}(\cdot)$ refers to the probability of an event. The graph \mathcal{G}_k represents the directed time-varying communication network topology at the k^{th} time instant. Let \mathcal{N}_k^j denote the neighbors of the j^{th} agent at the k^{th} time instant from which it receives information. The set of inclusive neighbors of the j^{th} agent is denoted by $\mathcal{J}_k^j := \mathcal{N}_k^j \cup \{j\}$. Let \mathbb{N} , \mathbb{Z}^* , and \mathbb{R} be the sets of natural numbers (positive integers), nonnegative integers, and real numbers respectively. Let σ represent the singular value of a matrix. Let $\text{diag}(\alpha)$ denote the diagonal matrix of appropriate size with α as its diagonal elements. Let \min^+ refer to the minimum of the positive elements. Let $\mathbf{1} = [1, 1, \dots, 1]^T$, \mathbf{I} , $\mathbf{0}$, and \emptyset be the ones (column) vector, the identity matrix, the zero matrix of appropriate sizes, and the empty set respectively. Let $\|\cdot\|_p$ denote the ℓ_p vector norm. The symbols $|\cdot|$ and $\lceil \cdot \rceil$ denote the absolute value or the cardinality of a set and the ceiling function respectively.

II. PROBLEM STATEMENT AND OVERVIEW OF PSG-IMC

Let $\mathcal{R} \subset \mathbb{R}^{n_x}$ denote the n_x -dimensional compact physical space over which the swarm is distributed. The region \mathcal{R} is partitioned into n_{cell} disjoint bins represented by $R[i]$, $i = 1, \dots, n_{\text{cell}}$ so that $\bigcup_{i=1}^{n_{\text{cell}}} R[i] = \mathcal{R}$ and $R[i] \cap R[q] = \emptyset$, if $i \neq q$.

Let $m \in \mathbb{N}$ agents belong to the swarm. Note that we assume $m \gg n_{\text{cell}}$, since we deal with the swarm density distribution over these bins. Let the row vector \mathbf{r}_k^j represent the bin in which the j^{th} agent is actually present at the k^{th} time instant. If $\mathbf{r}_k^j[i] = 1$, then the j^{th} agent is inside the $R[i]$ bin at the k^{th} time instant; otherwise $\mathbf{r}_k^j[i] = 0$. The current swarm distribution (\mathcal{F}_k^*) is given by the ensemble mean of actual agent positions, i.e., $\mathcal{F}_k^* := \frac{1}{m} \sum_{j=1}^m \mathbf{r}_k^j$. Let us now define the desired formation.

Definition 1. (Desired Formation π) Let the desired formation shape be represented by a probability (row) vector $\pi \in \mathbb{R}^{n_{\text{cell}}}$ over the bins in \mathcal{R} , i.e., $\pi \mathbf{1} = 1$. Note that π can be any arbitrary probability vector, but it is the same for all agents within the swarm. In the presence of motion constraints, π needs to satisfy Assumption 3 discussed in Section VI. For example, π is the shape of the letter "I" in Fig. 1. Note that any guidance algorithm, using a swarm of m agents, can only achieve the best quantized representation of the desired formation π , where $\frac{1}{m}$ is the quantization factor. \square

The objectives of probabilistic swarm guidance using inhomogeneous Markov chains (PSG-IMC) running on board each agent are as follows:

- 1) Determine the agent's trajectory using a Markov chain, which obeys motion constraints, so that the overall swarm converges to a desired formation (π).
- 2) Reduce the number of transitions for achieving and maintaining the formation in order to reduce control effort (e.g., fuel).
- 3) Maintain the swarm distribution and automatically detect and repair damages to the formation.

The key idea of the proposed PSG-IMC is to synthesize inhomogeneous Markov chains for each agent so that each agent can independently determine its trajectory while the swarm distribution converges to the desired formation. The pseudo code for the algorithm is given in **Algorithm 1**.

Algorithm 1 Probabilistic swarm guidance algorithm using inhomogeneous Markov chains (PSG-IMC)

- | | |
|---|--|
| <pre> 1: (one cycle of j^{th} agent during k^{th} time instant) 2: Agent determines its present bin, e.g., $\mathbf{r}_k^j[i] = 1$ 3: Set m_{loop}, the weighting factors $a_k^{j\ell}$ 4: for $\nu = 1$ to m_{loop} 5: if $\nu = 1$ then Set $\mathcal{F}_{k,0}^j$ from \mathbf{r}_k^j end if 6: Transmit the pmf $\mathcal{F}_{k,\nu-1}^j$ to other agents 7: Obtain the pmfs $\mathcal{F}_{k,\nu-1}^\ell, \forall \ell \in \mathcal{J}_k^j$ 8: Compute the new pmf $\mathcal{F}_{k,\nu}^j$ using LinOP 9: end for 10: Compute the tuning parameter ξ_k^j 11: if $R[i] \notin \Pi$ then Go to bin $R[\ell]$ 12: else Compute the α_k^j vector 13: Compute the Markov matrix M_k^j 14: Modify the Markov matrix \tilde{M}_k^j 15: Generate a random number $z \in \text{unif}[0; 1]$ 16: Go to bin $R[q]$ such that $\sum_{\ell=1}^{q-1} \tilde{M}_k^j[i, \ell] \leq z < \sum_{\ell=1}^q \tilde{M}_k^j[i, \ell]$ 17: end if </pre> | <div style="display: flex; align-items: center; justify-content: center;"> <div style="font-size: 3em; margin-right: 10px;">}</div> <div> <p>Theorem 1</p> <p>Consensus Stage</p> <p>Eq. (3)</p> <p>Eq. (19)</p> <p>Eq. (8)</p> <p>Prop. 2</p> <p>Prop. 6</p> <p>Random sampling</p> </div> </div> |
|---|--|
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The first step (line 2) involves determining the bin in which the agent is located. For example, $\mathbf{r}_k^j[i] = 1$. During the second step (lines 3–9), each agent estimates the current swarm distribution (\mathcal{F}_k^*) by reaching an agreement across the network using the consensus algorithm [31]. This is elucidated in Section III.

In this paper we use the difference between the estimated swarm distribution ($\hat{\mathcal{F}}_{k, m_{\text{loop}}}^j$) and the desired formation (π) to dictate the motion of agents in the swarm. The Hellinger distance (HD) is a symmetric measure of the difference between two probability distributions and it is upper bounded by 1 [34]. As discussed in Section III, the tuning parameter (ξ_k^j) computed in line 10 is the HD between the current swarm distribution and the desired formation.

It may be desired that the agents in a particular bin can only transition to some bins, while they cannot transition to other bins. Each agent checks if it is currently in the trapping region or a transient bin in line 11 and then it transitions to the bin $R[\ell]$ which is best-suited to reach the formation. The concept of trapping region and the method for handling motion constraints are presented in Section VI.

The next step (lines 12–14) involves designing a family of

row stochastic Markov transition matrices \tilde{M}_k^j with π as their stationary distributions, which is presented in Sections IV and VI. When the HD between the estimated swarm distribution and the desired formation is large, each agent propagates its position in a statistically-independent manner so that the swarm asymptotically tends to the desired formation. As this HD decreases, the Markov matrices also tend to an identity matrix and each agent holds its own position. In lines 15–16, random sampling of the Markov matrix generates the next location of the agent. The stability and convergence guarantees for PSG-IMC are presented in Section V.

The guidance or motion planning of spacecraft in a swarm is discussed in Section VII. Strategies for implementing PSG-IMC algorithms are demonstrated with numerical examples in Sections VI and VII.

III. CONSENSUS ESTIMATION OF SWARM DISTRIBUTION

In this section, we use the decentralized consensus algorithm [31] to estimate the current swarm distribution, as illustrated in **Algorithm 1**. The objective of the consensus stage is to estimate the current swarm distribution (\mathcal{F}_k^*) and maintain consensus across the network during each time step.

Let the row vector $\hat{\mathcal{F}}_{k,\nu}^j$ represent the j^{th} agent's estimate of the current swarm distribution during the ν^{th} consensus loop at the k^{th} time instant. At the beginning of the consensus stage of the k^{th} time instant, the j^{th} agent generates a row vector of local estimate of the swarm distribution $\hat{\mathcal{F}}_{k,0}^j$ by only determining its present bin location:

$$\hat{\mathcal{F}}_{k,0}^j[i] = 1 \text{ if } \mathbf{r}_k^j[i] = 1, \text{ otherwise } 0. \quad (1)$$

In essence, the local estimate at the start of the consensus stage is a discrete representation of the position of the j^{th} agent in the space \mathcal{R} , i.e., $\hat{\mathcal{F}}_{k,0}^j = \mathbf{r}_k^j$. Hence the current swarm distribution is also given by $\mathcal{F}_k^* = \sum_{i=1}^m \frac{1}{m} \hat{\mathcal{F}}_{k,0}^i$, which is equal to the ensemble mean of actual agent positions $\{\mathbf{r}_k^j\}_{j=1}^m$ over the bins in \mathcal{R} .

During the consensus stage, the agents recursively combine and update their local distributions to reach an agreement across the network. The Linear Opinion Pool (LinOP) of probability measures, which is used for combining individual distributions [35], [36], is given by:

$$\hat{\mathcal{F}}_{k,\nu}^j = \sum_{\ell \in \mathcal{J}_k^j} a_{k,\nu-1}^{\ell j} \hat{\mathcal{F}}_{k,\nu-1}^{\ell}, \forall j, \ell \in \{1, \dots, m\}, \forall \nu \in \mathbb{N}, \quad (2)$$

where $\sum_{\ell \in \mathcal{J}_k^j} a_{k,\nu-1}^{\ell j} = 1$. The updated distribution $\hat{\mathcal{F}}_{k,\nu}^j$ after the ν^{th} consensus loop is a weighted average of the distributions of the inclusive neighbors $\hat{\mathcal{F}}_{k,\nu-1}^{\ell}, \forall \ell \in \mathcal{J}_k^j$ at k^{th} time instant. Let $\mathcal{W}_{k,\nu} = [\hat{\mathcal{F}}_{k,\nu-1}^1, \dots, \hat{\mathcal{F}}_{k,\nu-1}^m]$ be a row vector of pmf functions of the agents after the ν^{th} consensus loop. The LinOP (2) can be expressed concisely as $\mathcal{W}_{k,\nu} = \mathcal{W}_{k,\nu-1} P_{k,\nu-1}$, $\forall \nu \in \mathbb{N}$, where $P_{k,\nu-1}$ is a matrix with entries $P_{k,\nu-1}[\ell, j] = a_{k,\nu-1}^{\ell j}$.

Assumption 1. The communication network topology of the multi-agent system $\mathcal{G}(k)$ is strongly connected (SC). The weighting factors $a_{k,\nu-1}^{\ell j}, \forall j, \ell \in \{1, \dots, m\}$ and the matrix

$P_{k,\nu-1}$ have the following properties: (i) the weighting factors are the same for all consensus loops within each time instants; (ii) the matrix P_k conforms with the graph $\mathcal{G}(k)$; (iii) the matrix P_k is column stochastic; and (iv) the weighting factors $a_{k,\nu-1}^{\ell j}$ are balanced. \square

Since $m \gg n_{\text{cell}}$ and multiple agents are within the same bin, it is guaranteed almost surely using Erdős-Rényi random graphs or random nearest-neighbor graphs that the resulting network is SC [37], [38], [39]. Moreover, distributed algorithms exist for the agents to generate SC balanced graphs [28], [40], [41].

Let $\boldsymbol{\theta}_{k,\nu} = [\theta_{k,\nu}^1, \dots, \theta_{k,\nu}^m]$ be the disagreement vector, where $\theta_{k,\nu}^j$ is the \mathcal{L}_1 distances between $\hat{\mathcal{F}}_{k,\nu}^j$ and \mathcal{F}_k^* , i.e., $\theta_{k,\nu}^j = \sum_{R[i] \in \mathcal{R}} |\hat{\mathcal{F}}_{k,\nu}^j[i] - \mathcal{F}_k^*[i]|$. Since the \mathcal{L}_1 distances between pmfs is bounded by 2, the ℓ_2 vector norm $\|\boldsymbol{\theta}_{k,\nu}\|_2$ is upper bounded by $2\sqrt{m}$.

Theorem 1. [25]–[31] (Consensus using the LinOP on SC Balanced Digraphs) *Under Assumption 1, using the LinOP (2), each $\hat{\mathcal{F}}_{k,\nu}^j$ globally exponentially converges to $\mathcal{F}_k^* = \sum_{i=1}^m \frac{1}{m} \hat{\mathcal{F}}_{k,0}^i$ pointwise with a rate faster or equal to the second largest singular value of P_k , i.e., $\sigma_{m-1}(P_k)$. If for some consensus error $\varepsilon_{\text{cons}} > 0$, the number of consensus loops within each consensus stage is $n_{\text{loop}} \geq \left\lceil \frac{\ln(\varepsilon_{\text{cons}}/(2\sqrt{m}))}{\ln \sigma_{m-1}(P_k)} \right\rceil$ then the ℓ_2 norm of the disagreement vector at the end of the consensus stage is less than $\varepsilon_{\text{cons}}$, i.e., $\|\boldsymbol{\theta}_{k,n_{\text{loop}}}\|_2 \leq \varepsilon_{\text{cons}}$.*

In this paper, we assume $\varepsilon_{\text{cons}} \leq \frac{1}{m}$ and n_{loop} is computed using Theorem 1. Note that, in order to transmit the pmf $\hat{\mathcal{F}}_{k,\nu}^j$ to another agent, the agent needs to transmit n_{cell} real numbers bounded by $[0, 1]$. For a practical scenario involving spacecraft swarms, as discussed in Section VII, it is feasible to execute multiple consensus loops within each time step.¹

Next, we compute the tuning parameter (ξ_k^j) described in line 10 of **Algorithm 1**.

Definition 2. (Hellinger Distance based Tuning Parameter ξ_k^j) The HD is a symmetric measure of the difference between two probability distributions and it is upper bounded by 1 [34], [43]. Each agent chooses the tuning parameter (ξ_k^j), based on the HD, using the following equation:

$$\xi_k^j = D_H(\boldsymbol{\pi}, \hat{\mathcal{F}}_{k,n_{\text{loop}}}^j) := \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^{n_{\text{cell}}} \left(\sqrt{\boldsymbol{\pi}[i]} - \sqrt{\hat{\mathcal{F}}_{k,n_{\text{loop}}}^j[i]} \right)^2}. \quad (3)$$

where $\hat{\mathcal{F}}_{k,n_{\text{loop}}}^j$ is the estimated current swarm distribution and $\boldsymbol{\pi}$ is the desired formation. \square

Remark 3. In contrast with the standard \mathcal{L}_1 or \mathcal{L}_2 distance metrics, the weight assigned by the HD to the error in a bin is inversely proportional to the square root of the desired distribution in that bin. For example, the \mathcal{L}_1 distances of

¹In order to obtain an estimate of the communication load, let us assume that during each of the 20 consensus loops, each agent needs to transmit its estimated pmf $\hat{\mathcal{F}}_{k,\nu}^j$ to 100 other agents and receives pmf estimates from 100 neighboring agents. The total transmission time using a 250 Kbps XBee radio [42] is $\frac{900 \times 8 \times 2 \times 100 \times 20}{250 \times 10^3} \approx 2$ minutes. This is significantly less than the time step of 8–10 minutes used for such missions.

$\mathcal{F}_1 = [0.1, 0, 0.4, 0, 0.5]$ and $\mathcal{F}_2 = [0, 0, 0.5, 0, 0.5]$ from the desired distribution $\pi = [0, 0, 0.4, 0, 0.6]$ are equal to 0.2. However, the agents in the first bin with $\mathcal{F}_1[1] = 0.1$ should move to other bins with nonzero probability, which is better encapsulated by the Hellinger distance (e.g., $D_H(\pi, \mathcal{F}_1) = 0.2286$ while $D_H(\pi, \mathcal{F}_2) = 0.0712$). \square

IV. FAMILY OF MARKOV TRANSITION MATRICES

The key concept of PSG-IMC is that each agent can independently determine its trajectory from the evolution of an inhomogeneous Markov chain with a desired stationary distribution. In this section, we design the family of Markov transition matrices for a desired stationary distribution, using the tuning parameter from Definition 2.

Let $\mathbf{x}_k^j \in \mathbb{R}^{n_{\text{cell}}}$ denote the row vector of probability mass function (pmf) of the predicted position of the j^{th} agent at the k^{th} time instant, i.e., $\mathbf{x}_k^j \mathbf{1} = 1$. The i^{th} element ($\mathbf{x}_k^j[i]$) is the probability of the event that the j^{th} agent is in $R[i]$ bin at the k^{th} time instant:

$$\mathbf{x}_k^j[i] = \mathbb{P}(\mathbf{r}_k^j[i] = 1), \quad \text{for all } i \in \{1, \dots, n_{\text{cell}}\}. \quad (4)$$

The elements of the row stochastic Markov transition matrix $M_k^j \in \mathbb{R}^{n_{\text{cell}} \times n_{\text{cell}}}$ are the transition probabilities of the j^{th} agent at the k^{th} time instant:

$$M_k^j[i, \ell] := \mathbb{P}(\mathbf{r}_{k+1}^j[\ell] = 1 | \mathbf{r}_k^j[i] = 1). \quad (5)$$

In other words, the probability that the j^{th} agent in $R[i]$ bin at the k^{th} time instant will transition to $R[\ell]$ bin at the $(k+1)^{\text{th}}$ time instant is given by $M_k^j[i, \ell]$. The Markov transition matrix M_k^j determines the time evolution of the pmf row vector \mathbf{x}_k^j by:

$$\mathbf{x}_{k+1}^j = \mathbf{x}_k^j M_k^j, \quad \text{for all } k \in \mathbb{Z}^*. \quad (6)$$

Lines 12–14 of **Algorithm 1** involves designing a family of Markov transition matrices for each agent M_k^j , with π as their stationary distributions. The following theorem is used by each agent to find these Markov matrices at each time instant.

Proposition 2. (Family of Markov transition matrices for a desired stationary distribution) Let $\alpha_k^j \in \mathbb{R}^{n_{\text{cell}}}$ be a nonnegative bounded column vector with $\|\alpha_k^j\|_\infty \leq 1$. For given ξ_k^j from (3), the following parametrized family of row stochastic Markov matrices M_k^j have π as their stationary distribution (i.e., $\pi M_k^j = \pi$):

$$M_k^j = \alpha_k^j \frac{\xi_k^j}{\pi \alpha_k^j} \pi \text{diag}(\alpha_k^j) + \mathbf{I} - \xi_k^j \text{diag}(\alpha_k^j), \quad (7)$$

where $\pi \alpha_k^j \neq 0$ and $\sup_k \xi_k^j \|\alpha_k^j\|_\infty \leq 1$.

Proof: For a valid first term in (7), we need $\pi \alpha_k^j \neq 0$. We first show that M_k^j is a row stochastic matrix. Right multiplying both sides of (7) with $\mathbf{1}$ gives:

$$M_k^j \mathbf{1} = \xi_k^j \alpha_k^j \frac{\pi \alpha_k^j}{\pi \alpha_k^j} + \mathbf{1} - \xi_k^j \text{diag}(\alpha_k^j) \mathbf{1} = \mathbf{1}.$$

Next, we show that M_k^j is a Markov matrix with π as its stationary distribution, as π is the left eigenvector corresponding

to its largest eigenvalue 1, i.e., $\pi M_k^j = \pi$. Left multiplying both sides of (7) with π gives:

$$\pi M_k^j = \frac{\pi \alpha_k^j}{\pi \alpha_k^j} \pi \xi_k^j \text{diag}(\alpha_k^j) + \pi - \pi \xi_k^j \text{diag}(\alpha_k^j) = \pi.$$

In order to ensure that all the elements in the matrix M_k^j are nonnegative, we enforce that $\mathbf{I} - \xi_k^j \text{diag}(\alpha_k^j) \geq 0$ which results in the condition $\sup_k \xi_k^j \|\alpha_k^j\|_\infty \leq 1$. \blacksquare

The additional degrees of freedom due to α_k^j vector allows us to capture the physical distance between bins while designing the Markov matrix.

Definition 4. (Physical distance based α_k^j vector) The j^{th} agent at the k^{th} time instant selects bin $R[c]$ independent of its current location. Then each element of the α_k^j vector is determined using the physical distance between bins in the following manner for all $\ell \in \{1, \dots, n_{\text{cell}}\}$:

$$\alpha_k^j[\ell] := 1 - \frac{\text{dis}(R[\ell], R[c])}{\max_{q \in \{1, \dots, n_{\text{cell}}\}} \text{dis}(R[q], R[c]) + 1}, \quad (8)$$

where $\text{dis}(R[\ell], R[c])$ is the ℓ_1 distance between the bins $R[\ell]$ and $R[c]$. If $\kappa[\ell] \in \mathbb{R}^{n_x}$ denotes the location of the centroid of the bin $R[\ell]$, then $\text{dis}(R[\ell], R[c]) = \|\kappa[\ell] - \kappa[c]\|_1$. Irrespective of the distribution of π , the condition $\pi \alpha_k^j \neq 0$ is satisfied because $\alpha_k^j[\ell] > 0, \forall \ell \in \{1, \dots, n_{\text{cell}}\}$. Moreover, the condition $\sup_k \xi_k^j \|\alpha_k^j\|_\infty \leq 1$ is satisfied as $\xi_k^j = D_H(\pi, \hat{\mathcal{F}}_{k, n_{\text{loop}}}^j) \leq 1$ in (3) and $\|\alpha_k^j\|_\infty = 1$ in (8). \square

The evolution of the agent's location during each time step is based on random sampling of the Markov transition matrix, as shown in line 15–16 of **Algorithm 1**. An agent is said to have undergone a transition if it jumps from bin $R[i]$ to bin $R[\ell]$, $\ell \neq i$ during a given time step. If $\xi_k^j = 0$ in (7), then $M_k^j = \mathbf{I}$ and no transitions occur as the transition probabilities from a bin to any bin (other than itself) goes to zero. If ξ_k^j is large (close to 1), the agents vigorously transition from one bin to another. In this paper, we seek to minimize unnecessary bin-to-bin transitions during each time step while maintaining or reconfiguring the formation.

Remark 5. (Dynamics based α_k^j vector) The agent dynamics can also be captured using the α_k^j vector. At the k^{th} time instant, the $\alpha_k^j[\ell]$ element is the likelihood of the j^{th} agent transitioning to the bin $R[\ell]$ due to its dynamics and is independent of its current location. In order to obtain valid solutions, it is necessary that $\alpha_k^j[\ell] > 0, \forall \ell \in \{1, \dots, n_{\text{cell}}\}$ and $\|\alpha_k^j\|_\infty = 1$ for all time instants. \square

In the probabilistic guidance algorithm (PGA) [17], [18], the same homogeneous Markov transition matrix is used by all agents for all time, i.e., $M_k^j = M$ for all $j \in \{1, \dots, m\}$ and $k \in \mathbb{Z}^*$. As the agents do not realize whether the desired formation has already been achieved, they continue to transition for all time steps. In contrast, in this paper, each agent executes a different time-varying Markov matrix M_k^j at each time step. We show that the inhomogeneous Markov chain not only guides the agents so that the swarm distribution converges to the desired steady-state distribution but also reduces the number of transitions across the bins.

V. CONVERGENCE ANALYSIS OF INHOMOGENEOUS MARKOV CHAINS

In this section, we provide asymptotic guarantees for the proposed PSG-IMC algorithm without motion constraints, and study its stability and convergence characteristics. Theorem 3 states that each agent's pmf vector \mathbf{x}_k^j asymptotically converges pointwise to the desired formation $\boldsymbol{\pi}$ while Theorem 4 states that the swarm distribution \mathcal{F}_k^* also asymptotically converges pointwise to the desired formation $\boldsymbol{\pi}$, when the number of agents tends to infinity. For practical implementation of this algorithm, a lower bound on the number of agents is provided in Theorem 5. Finally, Remark 6 states that if the consensus error tends to zero, then the Markov transition matrix M_k^j asymptotically converges to an identity matrix. This means that the agents stop moving after the desired formation is achieved, resulting in significant savings of control effort compared to the homogeneous Markov chain case. Therefore, the first two objectives of PSG-IMC, stated in Section II, are achieved.

The time evolution of the pmf vector \mathbf{x}_k^j , defined in (4), from an initial condition (\mathbf{x}_0^j) to the k^{th} time instant is given by the inhomogeneous Markov chain:

$$\mathbf{x}_k^j = \mathbf{x}_0^j U_{0,k}^j, \quad \text{for all } k \in \mathbb{N}, \quad (9)$$

where $U_{0,k}^j = M_0^j M_1^j \dots M_{k-2}^j M_{k-1}^j$ and each M_k^j is a $\mathbb{R}^{n_{\text{cell}} \times n_{\text{cell}}}$ row stochastic matrix obtained using Proposition 2.

We first focus on the convergence of each agent's pmf vector. In the following proof, we first show that the inhomogeneous Markov chain is strongly ergodic and then show that the limit is the desired distribution $\boldsymbol{\pi}$. Moreover, we assume that the swarm has not converged if at least two or more agents are not in the correct location. If at most one agent is not in its correct location,² then we state that the swarm has already converged and no further convergence is necessary.

Theorem 3. (Convergence of inhomogeneous Markov chains) *Each agent's time evolution of the pmf vector \mathbf{x}_k^j , from any initial condition $\mathbf{x}_0^j \in \mathbb{R}^{n_{\text{cell}}}$, is given by the inhomogeneous Markov chain (9). If each agent executes the PSG-IMC algorithm, then \mathbf{x}_k^j asymptotically converges pointwise to the desired stationary distribution $\boldsymbol{\pi}$, i.e., $\lim_{k \rightarrow \infty} \mathbf{x}_k^j = \boldsymbol{\pi}$ pointwise for all $j \in \{1, \dots, m\}$.*

Proof: According to Proposition 2, each matrix M_k^j is a function of the tuning parameter ξ_k^j which is determined by (3) and bounded by $[0, 1]$. Let us define the bins with nonzero probabilities in $\boldsymbol{\pi}$ as recurrent bins. According to the design of the Markov matrix, if $\xi_k^j > 0$, then an agent can enter these recurrent bins from all other bins. The bins which are not recurrent are called transient bins and they have zero probabilities in $\boldsymbol{\pi}$. The agents can only leave the transient bins when $\xi_k^j > 0$, but they can never enter a transient bin from any other bin due to the structure of the Markov matrix in (7). In this proof, we first consider the special case where all the bins are recurrent and show that each agent's pmf vector converges

to $\boldsymbol{\pi}$. We next consider the general case where transient bins are also present and show that this situation converges to the special case geometrically fast.

Case 1: all bins are recurrent, i.e., $\boldsymbol{\pi}[i] > 0$, for all $i \in \{1, \dots, n_{\text{cell}}\}$.

For $\xi_k^j > 0$, Proposition 2 guarantees that the matrix M_k^j is positive, strongly connected and primitive. We next prove that $\lim_{k \rightarrow \infty} U_{0,k}^j$ is also a primitive matrix. Forward multiplication of two row stochastic Markov matrices (M_s^j, M_{s+1}^j) with $\xi_s^j, \xi_{s+1}^j > 0$ and $s \in \mathbb{Z}^*$ obtained using Proposition 2 yields the positive matrix $M_s^j M_{s+1}^j$ which is a primitive matrix.

If $\xi_k^j = 0$, then $M_k^j = \mathbf{I}$ from (7). The matrix product $U_{0,k}^j$ can be decomposed into two parts; (a) the tuning parameter (ξ_s^j) of the matrices in the first part is always zero and (b) the second part contains the remaining Markov matrices with $\xi_s^j > 0$.

$$\begin{aligned} U_{0,k}^j &= M_0^j (\xi_0^j > 0) M_1^j (\xi_1^j > 0) M_2^j (\xi_2^j = 0) M_3^j (\xi_3^j > 0) \dots \\ &\quad M_{k-3}^j (\xi_{k-3}^j > 0) M_{k-2}^j (\xi_{k-2}^j = 0) M_{k-1}^j (\xi_{k-1}^j > 0), \\ &= \underbrace{(\mathbf{I} \dots \mathbf{I})}_{\text{1st part}} \cdot \underbrace{(M_0^j M_1^j M_3^j \dots M_{k-3}^j M_{k-1}^j)}_{\text{2nd part}}. \end{aligned}$$

Since the first part results in an identity matrix, the matrix product $U_{0,k}^j$ can be completely defined by the second part containing Markov matrices with $\xi_k^j > 0$. In the matrix product $\lim_{k \rightarrow \infty} U_{0,k}^j$, the tuning parameters (ξ_k^j) for the first few Markov matrices are nonzero because the swarm starts from initial conditions that are different from the desired formation. Hence there exists at least one matrix in the second part of $\lim_{k \rightarrow \infty} U_{0,k}^j$. Thus we can prove (by induction) that the matrix product $\lim_{k \rightarrow \infty} U_{0,k}^j$ is a primitive matrix.

Next, we prove that the matrix product $\lim_{k \rightarrow \infty} U_{0,k}^j$ is asymptotically homogeneous and strongly ergodic (See Definition 8 in Appendix). Let us find a positive γ independent of k such that $\gamma \leq \min^+ M_k^j[i, \ell]$ for all $i, \ell \in \{1, \dots, n_{\text{cell}}\}$. Let $\varepsilon_{\text{cons}} \leq \frac{1}{m}$ and n_{loop} is computed using Theorem 1. If at least two agents are not in the correct location, i.e., $\sum_{R[i] \in \mathcal{R}} |\mathcal{F}_k^*[i] - \boldsymbol{\pi}[i]| \geq \frac{2}{m}$; then due to the quantization of the pmf by the number of agents m , the smallest positive tuning parameter ξ_{\min} is given by:

$$\xi_{\min} = \frac{1}{2^{\frac{3}{2}} m} \leq \min_{j \in \{1, \dots, m\}, k \in \mathbb{Z}^*} \xi_k^j. \quad (10)$$

The smallest positive element in the $\boldsymbol{\alpha}_k^j$ vector in (8) is given by:

$$\alpha_{\min} = \left(1 - \frac{\max_{i, q \in \{1, \dots, n_{\text{cell}}\}} \text{dis}(R[q], R[i])}{\max_{i, q \in \{1, \dots, n_{\text{cell}}\}} \text{dis}(R[q], R[i]) + 1} \right). \quad (11)$$

Finally, the smallest positive element in the stationary distribution $\boldsymbol{\pi}$ is given by $\pi_{\min} = \left(\min_{i \in \{1, \dots, n_{\text{cell}}\}} \boldsymbol{\pi}[i] \right)$. The diagonal and off-diagonal elements of the Markov matrix M_k^j designed using (7) are given by:

$$M_k^j[i, i] = 1 - \xi_k^j \boldsymbol{\alpha}_k^j[i] + \frac{\xi_k^j}{\pi \boldsymbol{\alpha}_k^j} (\boldsymbol{\alpha}_k^j[i])^2 \boldsymbol{\pi}[i], \quad (12)$$

$$M_k^j[i, \ell] = \frac{\xi_k^j}{\pi \boldsymbol{\alpha}_k^j} \boldsymbol{\alpha}_k^j[i] \boldsymbol{\pi}[\ell] \boldsymbol{\alpha}_k^j[\ell], \quad \text{where } i \neq \ell. \quad (13)$$

Irrespective of the choice of ξ_k^j , the diagonal elements of M_k^j

²The probability of the event, that at most one agent is not in its correct location, is upper bounded by $(\frac{1}{n_{\text{rec}}})^{m-1}$. For the simulation example discussed in Section VI-A, it is approximately 10^{-7000} .

$$M_k^j = \begin{bmatrix} M[1,1] & \dots & M[1,n_{\text{rec}}] & M[1,n_{\text{rec}}+1] & \dots & M[1,n_{\text{cell}}] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M[n_{\text{rec}},1] & \dots & M[n_{\text{rec}},n_{\text{rec}}] & M[n_{\text{rec}},n_{\text{rec}}+1] & \dots & M[n_{\text{rec}},n_{\text{cell}}] \\ M[n_{\text{rec}}+1,1] & \dots & M[n_{\text{rec}}+1,n_{\text{rec}}] & M[n_{\text{rec}}+1,n_{\text{rec}}+1] & \dots & M[n_{\text{rec}}+1,n_{\text{cell}}] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M[n_{\text{cell}},1] & \dots & M[n_{\text{cell}},n_{\text{rec}}] & M[n_{\text{cell}},n_{\text{rec}}+1] & \dots & M[n_{\text{cell}},n_{\text{cell}}] \end{bmatrix}$$

$M_{k,\text{sub}}^j \in \mathbb{R}^{n_{\text{rec}} \times n_{\text{rec}}}$

Fig. 2. Submatrix $M_{k,\text{sub}}^j \in \mathbb{R}^{n_{\text{rec}} \times n_{\text{rec}}}$ of only the recurrent bins in the original Markov matrix M_k^j .

will not tend to zero. But the off-diagonal elements of M_k^j will tend to zero if $\xi_k^j \rightarrow 0$. Moreover, the largest possible value of $\pi \alpha_k^j = 1$, when $\alpha_k^j = 1$. Hence the smallest nonzero element in M_k^j is lower bounded by the smallest possible values of the terms in (13).

Thus we get a positive γ that is independent of k , i.e., $\gamma = \xi_{\min} \alpha_{\min}^2 \pi_{\min}$. If $\xi_k^j = 0$, then $\min_{i,\ell \in \{1,\dots,n_{\text{cell}}\}} M_k^j[i,\ell] = 1 > \gamma$. Since M_k^j is row stochastic, $M_k^j[i,\ell] \leq 1$ for all $i, \ell \in \{1, \dots, n_{\text{cell}}\}$ and $k \in \mathbb{Z}^*$. Thus each Markov matrix satisfies the condition (27) in Lemma 8 given in Appendix. Since $\psi = \pi$ in (26), it follows from Lemma 8 that the matrix product $\lim_{k \rightarrow \infty} U_{0,k}^j$ is asymptotically homogeneous and strongly ergodic.

Note that M_k^j for all $k \in \mathbb{Z}^*$, by Proposition 2, have π as their left eigenvector for the eigenvalue 1. Hence $\mathbf{v}_0^j = \pi$ is a set of absolute probability vectors for $\lim_{k \rightarrow \infty} U_{0,k}^j$. According to Lemma 9 given in Appendix, $\mathbf{v}_0^j = \pi$ is the unique set of absolute probability vectors for $\lim_{k \rightarrow \infty} U_{0,k}^j$ [44] and $\mathbf{w}_0^j = \pi$ in (25). Hence the individual pmfs asymptotically converge to:

$$\lim_{k \rightarrow \infty} \mathbf{x}_k^j = \lim_{k \rightarrow \infty} \mathbf{x}_0^j U_{0,k}^j = \mathbf{x}_0^j \mathbf{1} \pi = \pi. \quad (14)$$

Case 2: both transient and recurrent bins are present, i.e., $\pi[i] > 0$ for all $i \in \{1, \dots, n_{\text{rec}}\}$ and $\pi[i] = 0$ for all $i \in \{n_{\text{rec}}+1, \dots, n_{\text{cell}}\}$.

Without loss of generality, let us reorder the bins such that the first n_{rec} bins are recurrent, and the remaining bins are transient. Hence the pmf vector \mathbf{x}_k^j is also reordered so that $\mathbf{x}_k^j[i]$, $i \in \{1, \dots, n_{\text{rec}}\}$ is the probability of the event that the j^{th} agent is in the $R[i]$ recurrent bin at the k^{th} time instant. It is known that the agents leave the transient bins geometrically fast [23, Theorem 4.3, pp. 120]. Once all the agents are in the recurrent bins, then the situation is similar to Case 1 with n_{rec} bins. The submatrix $M_{k,\text{sub}}^j \in \mathbb{R}^{n_{\text{rec}} \times n_{\text{rec}}}$ of the original Markov matrix, as shown in Fig. 2, is primitive when $\xi_k^j > 0$. Then, it follows from Case 1 that $\lim_{k \rightarrow \infty} \mathbf{x}_k^j = \pi$. ■

Since each agent's pmf vector converges to π , we now focus on the convergence of the current swarm distribution.

Theorem 4. (Convergence of swarm distribution to desired formation) *If the number of agents, executing the PSG-IMC algorithm, tends to infinity; then the current swarm distribution ($\mathcal{F}_k^* = \frac{1}{m} \sum_{j=1}^m \mathbf{r}_k^j$) asymptotically converges pointwise to the desired stationary distribution π , i.e., $\lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \mathcal{F}_k^* = \pi$ pointwise.*

Proof: Let $X_k^j[i]$ denote the independent Bernoulli random variable representing the event that the j^{th} agent is actually located in bin $R[i]$ at the k^{th} time instant, i.e.,

$X_k^j[i] = 1$ if $\mathbf{r}_k^j[i] = 1$ and $X_k^j[i] = 0$ otherwise. Let $X_\infty^j[i]$ denote the random variable $\lim_{k \rightarrow \infty} X_k^j[i]$. Theorem 3 implies that the success probability of $X_\infty^j[i]$ is given by $\mathbb{P}(X_\infty^j[i] = 1) = \lim_{k \rightarrow \infty} \mathbf{x}_k^j[i] = \pi[i]$. Hence $\mathbb{E}[X_\infty^j[i]] = \pi[i] \cdot 1 + (1 - \pi[i]) \cdot 0 = \pi[i]$, where $\mathbb{E}[\cdot]$ is the expected value of the random variable. Let $S_\infty^m[i] = X_\infty^1[i] + \dots + X_\infty^m[i]$. As the random variables $X_\infty^j[i]$ are independent and identically distributed, the strong law of large numbers (cf. [45, pp. 85]) states that:

$$\mathbb{P}\left(\lim_{m \rightarrow \infty} \frac{S_\infty^m[i]}{m} = \pi[i]\right) = 1. \quad (15)$$

The final swarm distribution is also given by $\lim_{k \rightarrow \infty} \mathcal{F}_k^*[i] = \frac{1}{m} \sum_{j=1}^m \mathbf{r}_k^j[i] = \frac{S_\infty^m[i]}{m}$. Hence (15) implies that $\lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \mathcal{F}_k^* = \pi$ pointwise. It follows from Scheffé's theorem [46, pp. 84] that the measure induced by \mathcal{F}_k^* on the σ -algebra of \mathcal{R} converges in total variation to the measure induced by π . ■

In practical scenarios, the number of agents is finite, hence we need to specify a convergence error threshold. The following theorem gives the minimum number of agents needed to establish ε -convergence of the swarm.

Theorem 5. *For some acceptable convergence error $\varepsilon_{\text{conv}} > 0$ and $\varepsilon_{\text{bin}} > 0$, if the number of agents is at least $m \geq \frac{1}{4\varepsilon_{\text{bin}}^2 \varepsilon_{\text{conv}}}$, then the pointwise error probability for each bin is bounded by $\varepsilon_{\text{conv}}$, i.e., $\mathbb{P}\left(\left|\frac{S_\infty^m[i]}{m} - \pi[i]\right| > \varepsilon_{\text{bin}}\right) \leq \varepsilon_{\text{conv}}, \forall i \in \{1, \dots, n_{\text{cell}}\}$.*

Proof: The variance of the independent random variable from Theorem 4 is $\text{Var}(X_\infty^j[i]) = \pi[i](1 - \pi[i])$, hence $\text{Var}\left(\frac{S_\infty^m[i]}{m}\right) = \frac{\pi[i](1 - \pi[i])}{m}$. The Chebychev's inequality (cf. [46, Theorem 1.6.4, pp. 25]) implies that for any $\varepsilon_{\text{bin}} > 0$, the pointwise error probability for each bin is bounded by:

$$\mathbb{P}\left(\left|\frac{S_\infty^m[i]}{m} - \pi[i]\right| > \varepsilon_{\text{bin}}\right) \leq \frac{\pi[i](1 - \pi[i])}{m\varepsilon_{\text{bin}}^2} \leq \frac{1}{4m\varepsilon_{\text{bin}}^2}.$$

Hence, the minimum number of agents is given by $\frac{1}{4m\varepsilon_{\text{bin}}^2} \leq \varepsilon_{\text{conv}}$. ■

We now study the convergence of the tuning parameter and the Markov matrix.

Remark 6. (Convergence of Markov matrix to identity matrix) If the consensus error tends to zero, i.e., $\varepsilon_{\text{cons}} \rightarrow 0$, then $\hat{\mathcal{F}}_{k,n_{\text{loop}}}^j = \mathcal{F}_k^*$. If the number of agents, executing the PSG-IMC algorithm, tends to infinity then Theorem 4 implies that the limiting tuning parameter (3) is given by:

$$\begin{aligned} \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \lim_{n_{\text{loop}} \rightarrow \infty} \xi_k^j &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \lim_{n_{\text{loop}} \rightarrow \infty} D_H(\pi, \hat{\mathcal{F}}_{k,n_{\text{loop}}}^j) \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} D_H(\pi, \mathcal{F}_k^*) = D_H(\pi, \pi) = 0. \end{aligned}$$

Therefore Proposition 2 implies that the Markov transition matrix $\lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \lim_{n_{\text{loop}} \rightarrow \infty} M_k^j = \mathbf{I}$. □

In practical scenarios, n_{loop} is finite, hence ξ_k^j may not converge to zero. A practical work-around to avoid transitions due to nonzero ξ_k^j is to set $M_k^j = \mathbf{I}$ when ξ_k^j is sufficiently small and has leveled out. In the next section, we use the above theorems to prove convergence of the PSG-IMC algorithm with motion constraints.

VI. MOTION CONSTRAINTS

In this section, we introduce additional constraints on the motion of agents and study their effects on the convergence of the swarm. We first introduce the motion constraints and the corresponding trapping problem. Next, we discuss the strategy for leaving the trapping region and an additional condition on the desired formation. Finally, Theorem 7 shows that each agent's pmf vector \mathbf{x}_k^j asymptotically converges pointwise to the desired formation $\boldsymbol{\pi}$, if it executes the PSG-IMC algorithm with motion constraints.

The agents in a particular bin can only transition to some bins but cannot transition to other bins because of the dynamics or physical constraints. These (possibly time-varying) motion constraints are specified in a matrix A_k^j as follows:

$$A_k^j[i, \ell] = \begin{cases} 1 & \text{if the } j^{\text{th}} \text{ agent can transition to } R[\ell] \\ 0 & \text{if the } j^{\text{th}} \text{ agent cannot transition to } R[\ell] \end{cases},$$

where $\mathbf{r}_k^j[i] = 1$, for all $\ell \in \{1, \dots, n_{\text{cell}}\}$. (16)

Assumption 2. The matrix A_k^j is symmetric and the graph conforming to the A_k^j matrix is strongly connected. Moreover, an agent can always choose to remain in its present bin, i.e., $A_k^j[i, i] = 1$ for all $i \in \{1, \dots, n_{\text{cell}}\}$ and $k \in \mathbb{Z}^*$. \square

In this paper, we introduce a simple, intuitive method for handling motion constraints, such that the convergence results in Section V are not affected. The key idea is to modify the Markov matrix M_k^j designed in Proposition 2 to capture the motion constraints in (16).

Proposition 6. Let \tilde{M}_k^j represent the modified Markov matrix that satisfies motion constraints, which is obtained from the original Markov matrix M_k^j given by Proposition 2. For each transition that is not allowed by the motion constraint, i.e., $A_k^j[i, \ell] = 0$, the corresponding transition probability in M_k^j is added to the diagonal element in \tilde{M}_k^j . First, let $\tilde{M}_k^j = M_k^j$. Then set:

$$\tilde{M}_k^j[i, i] = M_k^j[i, i] + \sum_{\ell \in \{1, \dots, n_{\text{cell}} : A_k^j[i, \ell] = 0\}} M_k^j[i, \ell].$$

Finally, for all $i, \ell \in \{1, \dots, n_{\text{cell}}\}$:

$$\text{if } A_k^j[i, \ell] = 0, \text{ then set } \tilde{M}_k^j[i, \ell] = 0.$$

The resulting row stochastic Markov matrix \tilde{M}_k^j has $\boldsymbol{\pi}$ as its stationary distribution (i.e., $\boldsymbol{\pi} \tilde{M}_k^j = \boldsymbol{\pi}$).

Proof: The modified Markov matrix \tilde{M}_k^j is row stochastic since $\tilde{M}_k^j \mathbf{1} = M_k^j \mathbf{1} = \mathbf{1}$. Moreover, \tilde{M}_k^j also has $\boldsymbol{\pi}$ as its stationary distribution because of the reversible property of M_k^j , i.e. $\boldsymbol{\pi}[\ell] M_k^j[\ell, i] = \boldsymbol{\pi}[i] M_k^j[i, \ell]$. Hence \tilde{M}_k^j is indeed a valid Markov matrix. \blacksquare

For a bin $R[i]$, let us define $\mathcal{A}_k^j(R[i])$ as the set of all bins that the j^{th} agent can transition to at the k^{th} time instant:

$$\mathcal{A}_k^j(R[i]) := \{R[\ell] : \ell \in \{1, \dots, n_{\text{cell}}\}; A_k^j[i, \ell] = 1\}. \quad (17)$$

Similarly, let us define Π as the set of all bins that have nonzero probabilities in the desired formation ($\boldsymbol{\pi}$):

$$\Pi := \{R[\ell] : \ell \in \{1, \dots, n_{\text{cell}}\}; \text{ and } \boldsymbol{\pi}[\ell] > 0\}. \quad (18)$$

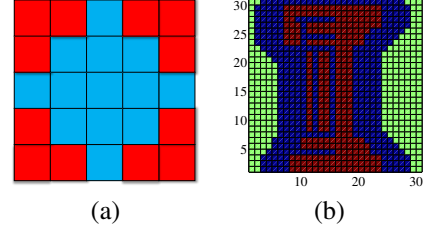


Fig. 3. (a) The red region denotes the set Π . The motion constraints are such that the agents can only transition to the immediate neighboring bins. Note that each of the four corners form four subsets of Π , such that any agent cannot transition from one subset to another subset without exiting the set Π . (b) The UIUC logo, marked in red, is the desired formation $\boldsymbol{\pi}$. Although \mathcal{R} is partitioned into 900 bins, the set Π contains only 262 bins. The trapping region \mathcal{T}_k^j for all $k \in \mathbb{N}$ is marked in green.

As defined before in the proof of Theorem 3, the bins in the set Π are called recurrent bins while those not in Π are called transient bins.

Remark 7. (Trapping Problem) If the j^{th} agent is actually located in bin $R[i]$ and we observe that $\mathcal{A}_k^j(R[i]) \cap \Pi = \emptyset$ for all $k \in \mathbb{Z}^*$, then the j^{th} agent is trapped in the bin $R[i]$ forever. Let us define \mathcal{T}_k^j as the set of all bins that satisfy this trapping condition:

$$\mathcal{T}_k^j := \bigcup_{i \in \{1, \dots, n_{\text{cell}}\}} \{R[i] : \mathcal{A}_k^j(R[i]) \cap \Pi = \emptyset\}. \quad (19)$$

To avoid this trapping problem, we enforce a secondary condition on the j^{th} agent if it is actually located in a bin belonging to the set \mathcal{T}_k^j . For each bin $R[i] \in \mathcal{T}_k^j$, we chose another bin $\Psi(R[i])$; where $\Psi(R[i]) \in \mathcal{A}_k^j(R[i])$ and either $\Psi(R[i]) \notin \mathcal{T}_k^j$ or $\Psi(R[i])$ is close to other bins which are not in \mathcal{T}_k^j . The secondary condition on the j^{th} agent is that it will transition to bin $\Psi(R[i])$ during the k^{th} time step. Moreover, the agents in the transient bins, which are not in the trapping region, eventually transition to the recurrent bins. We also speed up this process. The information about exiting the trapping region and the transient bins is captured by the matrix $C_k^j \in \mathbb{R}^{n_{\text{cell}} \times n_{\text{cell}}}$ given by:

$$C_k^j[i, \ell] = \begin{cases} 1 & \text{if } R[i] \in \mathcal{T}_k^j \text{ and } R[\ell] = \Psi(R[i]) \\ \frac{1}{|\mathcal{A}_k^j(R[i]) \cap \Pi|} & \text{else if } R[i] \notin \mathcal{T}_k^j \text{ and } R[i] \notin \Pi \\ & \text{and } R[\ell] \in \mathcal{A}_k^j(R[i]) \cap \Pi \\ 0 & \text{otherwise} \end{cases},$$

for all $i, \ell \in \{1, \dots, n_{\text{cell}}\}$, (20)

which is used instead of the Markov matrix in (6). Note that this secondary condition would not cause an infinite loop as the graph conforming to the A_k^j matrix is strongly connected. \square

It is possible that the set of all the bins that have nonzero probabilities in the desired formation (Π) can be decomposed into subsets, such that any agent cannot transition from one subset to another subset without exiting the set Π (for example see Fig. 3(a)). Since our proposed algorithm suggests that the agents will always transition within Π after entering it, the agents in one such subset will never transition to the other subsets. Hence, the proposed algorithms will not be able to achieve the desired formation, as the agents in each subset of Π are trapped within that subset. In order to avoid such

situations, we need the following assumption on Π and A_k^j .

Assumption 3. The set of all bins that have nonzero probabilities in the desired formation (Π) and the matrix specifying the motion constraints (A_k^j) are such that each agent can transition from any bin in Π to any other bin in Π , without exiting the set Π while satisfying the motion constraints. \square

For the j^{th} agent in bin $R[i]$, random sampling of the Markov chain selects the bin $R[q]$ in line 15–16 of **Algorithm 1**. Then the time evolution of the pmf vector \mathbf{x}_k^j can be written as:

$$\mathbf{x}_{k+1}^j = \mathbf{x}_k^j B_k^j, \text{ where } B_k^j = \begin{cases} C_k^j & \text{if } R[i] \notin \Pi \\ \tilde{M}_k^j & \text{otherwise} \end{cases}. \quad (21)$$

Note that the Markov matrix \tilde{M}_k^j in (21) is computed using Proposition 2 and then modified using Proposition 6. Let $V_{0,k}^j$ denote the row stochastic matrix defined by the forward matrix multiplication:

$$V_{0,k}^j = B_0^j B_1^j \dots B_{k-2}^j B_{k-1}^j, \quad \text{for all } k \in \mathbb{N}, \quad (22)$$

where each B_k^j is given by (21). Similar to (9), the evolution of the probability vector \mathbf{x}_k^j from an initial condition to any k^{th} time instant is given by $\mathbf{x}_k^j = \mathbf{x}_0^j V_{0,k}^j$ for all $k \in \mathbb{N}$.

Theorem 7. (Convergence of inhomogeneous Markov chains with motion constraints) Under Assumptions 2 and 3, each agent's time evolution of the pmf vector \mathbf{x}_k^j , from any initial condition $\mathbf{x}_0^j \in \mathbb{R}^{n_{\text{cell}}}$, is given by the inhomogeneous Markov chain $\mathbf{x}_k^j = \mathbf{x}_0^j V_{0,k}^j$. If each agent executes the PSG-IMC algorithm, then \mathbf{x}_k^j asymptotically converges pointwise to the desired stationary distribution π , i.e., $\lim_{k \rightarrow \infty} \mathbf{x}_k^j = \pi$ pointwise for all $j \in \{1, \dots, m\}$.

Proof: We first show that there are finitely many occurrences of the C_k^j matrix in the matrix product $\lim_{k \rightarrow \infty} V_{0,k}^j$. Due to the design of the Markov matrix in Proposition 2, the bins in the set Π are absorbing; i.e., if an agent enters any of the bins in the set Π , then it cannot leave the set Π . Next we notice that if $R[\ell]$ is a transient bin but not in the trapping region, then the only possible transitions are to the bins in Π . Moreover, once the agent exits the set \mathcal{T}_k^j , it cannot enter it again. Finally, the number of steps inside the transient bins is limited by the total number of bins n_{cell} . Hence the C_k^j can only occur finite number of times in the matrix product $\lim_{k \rightarrow \infty} V_{0,k}^j$. The new initial condition of the agent is obtained by forward multiplying the previous initial condition with the C_k^j matrices:

$$\tilde{\mathbf{x}}_0^j = \mathbf{x}_0^j C_0^j C_1^j \dots C_{s-1}^j C_s^j, \quad (23)$$

where s is the maximum number of steps that the j^{th} agent makes in the transient bins. Hence the overall time evolution of the pmf \mathbf{x}_k^j can be written as $\lim_{k \rightarrow \infty} \mathbf{x}_k^j = \lim_{k \rightarrow \infty} \mathbf{x}_0^j V_{0,k}^j = \lim_{k \rightarrow \infty} \tilde{\mathbf{x}}_0^j U_{0,k}^j$. Here the matrix product $\lim_{k \rightarrow \infty} U_{0,k}^j$ is the product of modified Markov matrices obtained using Proposition 6.

Once the agent exits the transient bins, the situation is exactly similar to that discussed in Case 1 of the proof of Theorem 3. Hence, we now focus on proving that the submatrix $\tilde{M}_{k,\text{sub}}^j \in \mathbb{R}^{n_{\text{rec}} \times n_{\text{rec}}}$ of the original Markov matrix \tilde{M}_k^j , similar

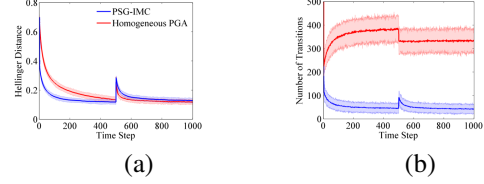


Fig. 4. The PSG-IMC algorithm is compared with the homogeneous PGA algorithm using Monte Carlo simulations. The figures show (a) convergence of the swarm to the desired formation in terms of HD and (b) the number of agents transitioning per time step, along with their 3σ error bars.

to that shown in Fig. 2, is primitive when $\xi_k^j > 0$. The modified Markov submatrix $\tilde{M}_{k,\text{sub}}^j$ is a nonnegative matrix due to the motion constraints A_k^j . But $\tilde{M}_{k,\text{sub}}^j$ is strongly connected and irreducible when $\xi_k^j > 0$ due to Assumptions 2 and 3. Since $(\tilde{M}_{k,\text{sub}}^j)^{n_{\text{rec}}} > 0$, the primitive matrix theorem [47, Theorem 8.5.2, pp. 516] implies that $\tilde{M}_{k,\text{sub}}^j$ is a primitive matrix. Then it follows from Case 1 that the product of submatrices $\lim_{k \rightarrow \infty} U_{0,k,\text{sub}}^j$ is also a primitive matrix. Finally, it follows from the proof of Theorem 3 that $\lim_{k \rightarrow \infty} \mathbf{x}_k^j = \pi$ pointwise for all $j \in \{1, \dots, m\}$. \blacksquare

Note that Theorem 4 and Remark 6 can be directly applied to satisfy the first two objectives of PSG-IMC, even under motion constraints.

A. Numerical Example

In this example, the PSG-IMC algorithm with motion constraints is used to guide a swarm containing $m = 3000$ agents to the desired formation π associated with the UIUC logo shown in Fig. 3(b). Monte Carlo simulations were performed to compare the PSG-IMC algorithm with the homogeneous PGA algorithm and the cumulative results from 50 runs are shown in Fig. 4. As shown in Fig. 5, at $k = 1$, each simulation starts with the agents uniformly distributed across $\mathcal{R} \subset \mathbb{R}^2$, which is partitioned into 30×30 bins. Each agent independently executes the PSG-IMC with motion constraints, illustrated in **Algorithm 1**. During the consensus stage, each agent is allowed to communicate with those agents which are at most 10 steps away. The communication matrix P_k is generated using Metropolis weights [28], hence it satisfies Assumption 1. Moreover, each agent is allowed to transition to only those bins which are at most 5 steps away.³

As shown in the HD graph in Fig. 4(a), the desired formation is almost achieved within the first 200 time steps for all simulation runs. After the 500th time step, the swarm is externally damaged by eliminating approximately 460 ± 60 agents from the middle section of the formation. This can be seen by comparing the images for the 500th and 501st time step in Fig. 5 and the discontinuity during the 501st time step in Fig. 4. Note that the swarm always recovers from this damage and attains the desired formation within another 100 time steps. Thus the third objective of PSG-IMC, stated in Section II, is also achieved. From the correlation between the plots of HD and number of transitions in Fig. 4(b), we can infer that the agents transition only when necessary.

³This simulation video is available at <http://youtu.be/KFFCYHgLfVw>.

During the first 500 time steps, each agent in the PSG-IMC algorithm undergoes just 10 transitions, compared to approximately 60 transitions in the homogeneous PGA algorithm. Moreover, the swarm of agents executing the PSG-IMC algorithm converges faster to the desired formation during this stage as seen in Fig. 4(a). Hence, it is evident from the cumulative results in Fig. 4 that the PSG-IMC algorithm is more efficient than the homogeneous PGA algorithm.

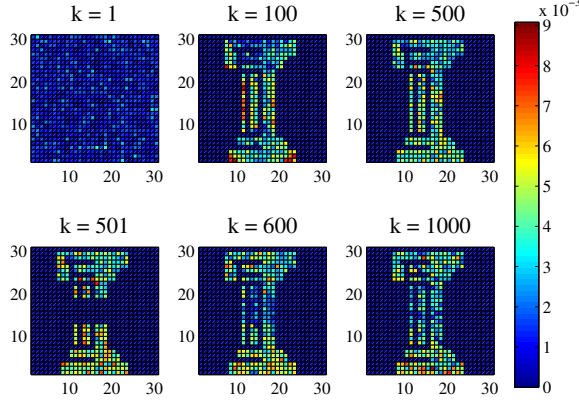


Fig. 5. Histogram plots of the swarm distribution at different time instants for 3000 agents, in a sample run of the Monte Carlo simulation. The colorbar represents the pmf of the swarm distribution. Each agent is allowed to transition to only those bins which are at most 5 steps away.

VII. GUIDANCE OF SPACECRAFT SWARMS

Since it is technically feasible to develop and deploy swarms (100s–1000s) of femtosatellites [1], in this section we solve the guidance problem for such swarms of spacecraft in Earth orbit.

Assume that the spacecraft are located in the Local Vertical Local Horizontal (LVLH) frame rotating around Earth. As shown in Fig. 6(a), for some initial conditions and no further control input, each spacecraft is in a closed elliptical orbit in the LVLH frame, called passive relative orbit (PRO) [48]. If time-varying bins ($R_k[i]$, $\forall i \in \{1, \dots, n_{\text{cell}}\}$) are designed so that the spacecraft continues to coast along the PRO in the LVLH frame, as shown in Fig. 6(a), then no control input is required to transition from $R_k[i]$ to $R_{k+1}[i]$.

Motion constraints could arise from the spacecraft dynamics, as it might be infeasible for the spacecraft in a certain bin to transition to another bin within a single time step due to the large distance between these bins. Some constraints could also arise from a limit on the amount of fuel that can be consumed in a single time step or limited control authority. A time-varying motion constraints matrix A_k^j is designed to handle such motion constraints due to spacecraft dynamics. If a spacecraft is currently in $R_k[i]$ bin, then it can only transition to the light blue cells in Fig. 6(b) during the $(k+1)$ th time instant due to the motion constraints. We have already shown that if each spacecraft executes the PSG-IMC algorithm, then each spacecraft satisfies these motion constraints, the swarm converges to the desired formation, and the Markov matrices converge to the identity matrix.

We now extend the example discussed in Section VI-A to a swarm of spacecraft in Earth orbit. If $(x[i], y[i])$ denotes the

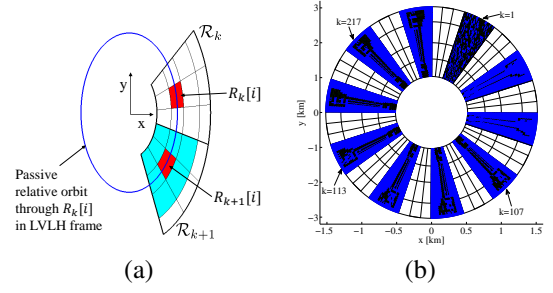


Fig. 6. (a) The PRO through $R_k[i]$ bin and the time-varying bin locations in the LVLH frame. (b) Location of the bins at different time steps in the LVLH frame along with the respective swarm distributions.

location of the bin $R[i]$ in the 30×30 grid, then the time-varying location of the centroid of the bin on the PRO in the LVLH frame is given by:

$$\kappa_k[i] = \begin{pmatrix} \frac{1}{2} \left(1 + \frac{1}{15} x[i] \right) \sin\left(\frac{\pi}{10} k + \frac{\pi}{300} y[i]\right) \\ \left(1 + \frac{1}{15} x[i] \right) \cos\left(\frac{\pi}{10} k + \frac{\pi}{300} y[i]\right) \end{pmatrix}. \quad (24)$$

Fig. 6(b) shows the locations of the time-varying bins at different time steps along with the respective swarm distributions. Similar to the previous example, the swarm converges to the desired formation and the spacecraft settle down after the final formation is achieved.

VIII. CONCLUSIONS

This paper presents a new approach to shaping and reconfiguring a large number of autonomous agents in a probabilistic framework. In the proposed PSG-IMC algorithm, each agent independently determines its own trajectory so that the overall swarm asymptotically converges to the desired formation, which is robust to external disturbances or damages. Compared to prior work using homogeneous Markov chains for all agents, the proposed algorithm using inhomogeneous Markov chains helps the agents avoid unnecessary transitions when the swarm converges to the desired formation. We present a novel technique for constructing inhomogeneous Markov matrices in a distributed fashion, thereby achieving and maintaining the desired swarm distribution while satisfying motion constraints. Using a consensus algorithm along with communication with neighboring agents, the agents estimate the current swarm distribution and transition so that the Hellinger distance between the estimated swarm distribution and the desired formation is minimized. The application of PSG-IMC algorithm to guide a swarm of spacecraft in Earth orbit is discussed. Results from multiple simulation runs demonstrate the properties of self-repair capability, reliability and improved convergence of the proposed PSG-IMC algorithm.

APPENDIX

In this section, we introduce some key definitions and lemmas that are used in the proof of Theorem 3.

Definition 8. (*Strong Ergodicity and Asymptotic Homogeneity*) The matrix product $U_{0,k}^j$ is defined to be strongly ergodic if [22]:

$$\lim_{k \rightarrow \infty} U_{0,k}^j = \mathbf{1} w_0^j, \quad \text{where } k \in \mathbb{N}, j \in \{1, \dots, m\} \quad (25)$$

where w_0^j is a row probability vector (i.e., $w_0^j \mathbf{1} = 1$).

A Markov chain (9) is defined to be asymptotically homogeneous (with respect to ψ) if there exists a row probability vector ψ (i.e., $\psi\mathbf{1} = 1$) such that [23, pp. 92]:

$$\lim_{k \rightarrow \infty} \psi M_k^j = \psi, \quad \text{where } k \in \mathbb{N}, j \in \{1, \dots, m\} \quad (26)$$

where $M_k^j \mathbf{1} = \mathbf{1}$ and $\psi \mathbf{1} = 1$. \square

Lemma 8. [23, pp. 97] (*Asymptotic Homogeneity implies Strong Ergodicity*) If the matrix product $\lim_{k \rightarrow \infty} U_{0,k}^j$ is a primitive matrix and there exists a positive γ independent of k such that the following condition holds $\forall k \in \mathbb{Z}^*$ and $\forall j \in \{1, \dots, m\}$:

$$0 < \gamma \leq \min_{i, \ell \in \{1, \dots, n_{\text{cell}}\}} M_k^j[i, \ell], \quad \max_{i, \ell \in \{1, \dots, n_{\text{cell}}\}} M_k^j[i, \ell] \leq 1, \quad \forall k \in \mathbb{Z}^*, \quad (27)$$

then asymptotic homogeneity of M_k^j is necessary and sufficient for strong ergodicity of $\lim_{k \rightarrow \infty} U_{0,k}^j$.

Lemma 9. [22] (*Uniqueness of Absolute Probability Vectors*) A set of absolute probability vectors for the matrix product $\lim_{k \rightarrow \infty} U_{0,k}^j$ is defined to be a sequence $\{\mathbf{v}_0^j\}$ of probability (row) vectors such that $\forall j \in \{1, \dots, m\}$:

$$\lim_{k \rightarrow \infty} \mathbf{v}_0^j U_{0,k}^j = \mathbf{v}_0^j, \quad \forall k \in \mathbb{N} \quad (28)$$

There is only one set of absolute probability vectors $\{\mathbf{v}_0^j\}$ if and only if the matrix product $\lim_{k \rightarrow \infty} U_{0,k}^j$ is strongly ergodic. In this case (25) holds with $\mathbf{w}_0^j = \mathbf{v}_0^j$.

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